Controllable Billiards: Characterizing the Paths of Simple Mobile Robots
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Motivation
- What tasks can simple robots perform?
- What are the minimal resources (sensing, actuation, computation time and space, feedback control) needed to complete tasks?
- By leveraging natural dynamics, make robots more efficient and robust!

Robotic Tasks as Properties of Paths
- **Navigation**: From a set of starting states, the robot’s path must end at the goal state(s).
- **Coverage**: The robot’s path must meet some coverage criterion (scan some fraction of environment).
- **Patrolling**: The path must be repeatable, and may also have some coverage criterion.

Approach
- Notice that many robots can travel forward in straight lines, identify when they’ve reached a boundary, and turn in place.
- Construct edge-to-edge mapping functions, compose and find fixed point.

Modeling One Transition
- \( x \): distance from the nearest clockwise vertex on edge of length \( \ell \).
- Each \( n \)-sided polygon and control angle \( \theta \) define system of piecewise linear transitions.
- \( x_{t+1} = b_\theta(x_t) = c(x_t - \ell) \)
- \( c = \frac{-\cos(\theta)}{\cos(\theta - \phi)} \)

Limit Cycles in Convex Polygons
- Compose edge-to-edge mappings to form \( B_\theta \), mapping one edge back to itself.
- In every convex \( n \)-sided polygon, there exists a range for \( \theta \) such that \( B_\theta \) has a fixed point (proof: show that \( B_\theta \) is a contraction map).

\[
x_{FP} = \left( \sum_{i=0}^{n-1} \ell_i \prod_{j=0}^{i-1} c_j \right) / \left( 1 - \prod_{k=0}^{n-1} c_k \right)
\]

Ongoing Work: Nonconvexity
- Can find "embedded" convex shapes by extending edges: number of potential limit cycles is exponential in number of sides.
- How many other limit cycles exist?

Ongoing Work: Coverage Properties
- Are trajectories ergodic?
- Density of contact with boundary over time?

Ongoing Work: Verification Tools
- Formal verification tools can augment analysis of hybrid dynamical systems.
- Can express stability as a reachability query: does trajectory ever leave initial set?
- Use refinement techniques to find minimal approximation to invariant set?

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